On asymptotically equivalent shallow water wave equations

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Abstract

The integrable third-order Korteweg–de Vries (KdV) equation emerges uniquely at linear order in the asymptotic expansion for unidirectional shallow water waves. However, at quadratic order, this asymptotic expansion produces an entire *family* of shallow water wave equations that are asymptotically equivalent to each other, under a group of nonlinear, non-local, normal-form transformations introduced by Kodama in combination with the application of the Helmholtz-operator. These Kodama–Helmholtz (KH) transformations are used to present connections between shallow water waves, the integrable fifth-order KdV equation, and a generalization of the Camassa–Holm (CH) equation that contains an additional integrable case. The dispersion relation of the full water wave problem and any equation in this family agree to fifth order. The travelling wave solutions of the CH equation are shown to agree to fifth order with the exact solution.
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1. Introduction

We study the irrotational incompressible flow of a shallow layer of inviscid fluid moving under the influence of gravity as well as surface tension. Previously, Dullin et al. [12] studied the case without surface tension, which in the shallow water approximation leads to the Camassa–Holm (CH) equation. CH is the following $1 + 1$ quadratically nonlinear equation for unidirectional water waves with fluid velocity $u(x, t)$,

$$m_t + c_0 m_x + u m_x + 2m u_x + \Gamma u_{xxx} = 0. \tag{1}$$

Here $m = u - \alpha^2 u_{xx}$ is a momentum variable, partial derivatives are denoted by subscripts, the constants α^2 and Γ/c_0 are squares of length scales and $c_0 = \sqrt{gh}$ is the linear wave speed for undisturbed water at rest at spatial infinity, where u and m are taken to vanish. The limit $\alpha^2 \to 0$ recovers the Korteweg–de Vries (KdV) equation [26].

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Eq. (1) was first derived by using asymptotic expansions directly in the Hamiltonian for Euler's equations for inviscid incompressible flow in the shallow water regime. It was thereby shown to be bi-Hamiltonian and integrable by the inverse scattering transform in the work of Camassa and Holm [5]. Its periodic solutions were treated by Alber et al. (see [1,2] and references therein). Before [5], families of integrable equations similar to (1) were known to be derivable in the general context of hereditary symmetries by Fokas and Fuchssteiner [17]. However, Eq. (1) was not written explicitly, nor was it derived physically as a water wave equation and its solution properties were not studied before [5]. See [18] for an insightful discussion of how the integrable shallow water equation (1) relates to the mathematical theory of hereditary symmetries.

Eq. (1) was recently rederived as a shallow water equation by using asymptotic methods in three different approaches by Fokas and Liu [15], Dullin et al. [12] and also by Johnson [23]. These three derivations used different variants of the method of asymptotic expansions for shallow water waves in the absence of surface tension. We shall derive an entire family of shallow water wave equations that are asymptotically equivalent to Eq. (1) at quadratic order in the shallow water expansion parameters. This is one order beyond the linear asymptotic expansion for the KdV equation. The asymptotically equivalent shallow water wave equations at quadratic order in this family are related amongst themselves by a continuous group of non-local transformations of variables that was first introduced by Kodama [24,25].

Dullin et al. [12] focused on the integrability of Eq. (1) and its spectral properties. Its derivation from Euler's equation in the case without surface tension was briefly described. Here we present the necessary details of this calculation. The present derivation also adds surface tension. In view of the many papers which have appeared recently on weakly nonlinear shallow water models (i.e. [16,23]) we see our contribution to be the following. By combining a non-local Kodama transformation with the application of the smoothing Helmholtz operator, we derive several integrable water wave equations in the same asymptotically equivalent family. This family includes the CH equation (1), the fifth-order Korteweg–de Vries (KdV5) equation and the equation of Degasperis and Procesi [9] which was recently discovered to be integrable in [10]. All these integrable equations are then explicitly related to each other, again by means of a Kodama transformation. We clarify the differences among the previous derivations of these equations. The equations are discussed with respect to their linear dispersion properties.

In the context of water waves subject to surface tension, interest has recently focused on the KdV5 equation and its solitary wave solutions. See [11] for a review. For Bond numbers $0 < \sigma < 1/3$ it has been shown that these solutions are not true solitary waves which decay to zero at spatial infinity but instead they are *generalized solitary waves* which are characterized by exponentially small ripples on their tail. See for example [4] for more explanation. It has been shown by Lombardi [28] that these ripples are exponentially small in terms of $F - 1$ where $F = c/c_0$ is the Froude number. Numerical experiments by Champneys et al. [7] suggest that in the full nonlinear water wave problem there are no real solitary waves bifurcating for Bond numbers $0 < \sigma < 1/3$. For a rigorous proof, see [32]. For Bond numbers larger than $1/3$, one obtains depressions with negative velocity, rather than elevations with positive velocity.

One may ask whether yet another model equation is needed, if the more rigorous, exact, or numerical results described above are already available. Or in more general terms: Which is preferred? An exact solution of an approximate model equation, or an approximate solution of an exact equation? Our point of view is: If the added cost is small, why not take both? It might be useful to have an equation that gives, e.g., more accurate travelling waves than KdV, without the need to go to much more elaborate models. Although we shall obtain less information and accuracy than the sophisticated, beyond-all-order methods, we shall also pay less. Thus, one can improve the description of the shape and speed of the travelling wave without having to resort to these more complicated models. That they are still only an approximation to the true solution is, of course, taken for granted.

Our inclusion of surface tension has a similar motivation. Although Eq. (30) that we shall derive has some drawbacks concerning the global properties of its dispersion relation for large k, it still gives improved descriptions for small k and small σ. Moreover, the improved solutions are easily obtained and analyzed.

The Kodama–Helmholtz (KH) transformation developed here can be useful not only in the context of water waves, but also in the many areas of science where the KdV equation appears as a model.

Outline. Section 2 recalls the standard dynamics for the shallow water wave elevation following Whitham [33]. We then use an approach based on the Kodama transformation to derive Eq. (1) with surface tension in Section 3. Section 4 explores the transformations employed to derive water wave equations. We discuss the class of equations which may be related to each other via such transformation, and are, hence, asymptotically equivalent. As examples, we discuss the relations of Eq. (1) to KdV and other integrable equations. We particularly discuss the relations to KdV5, the fifth-order integrable equation in the KdV hierarchy, and to another integrable nonlinear equation recently proposed by Degasperis and Procesi [9] and discovered to be integrable in [10]. Section 5 compares the dispersion relation of (1) with that of the full water wave equation. Finally Section 6 shows that the travelling wave solutions of (1) agree to fourth order with the exact travelling waves solutions of the full water wave problem.

2. The η equation

Our derivation of Eq. (1) proceeds from the physical shallow water system along the lines of Whitham [33]. Consider water of depth $h = h_0 + \eta(x, t)$, where $z = -h_0$ is the flat bottom and h_0 the mean depth, so that $z = 0$ at the free surface in equilibrium. Denote by u_h and u_v the horizontal and vertical velocity components, respectively. The z-momentum equation is

$$\frac{\mathrm{D}u_v}{\mathrm{D}t} = -g - \frac{1}{\rho}\partial_z p \quad \text{with} \quad p = \tilde{\sigma}\frac{h_{xx}}{(1 + h_x^2)^{3/2}}, \tag{2}$$

where g is the constant of gravity and $\tilde{\sigma}$ the surface tension. At the free surface, the boundary condition is

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} = u_v, \quad \text{where} \ z = \eta. \tag{3}$$

The irrotational velocity $\mathbf{u}(x, z, t) = \nabla\varphi$ has horizontal and vertical velocity components $u_h = \varphi_x$ and $u_v = \varphi_z$. The velocity potential φ must satisfy Laplace's equation in the interior. Eq. (3) yields the kinematic boundary condition at the free surface,

$$\eta_t + \varphi_x\eta_x = \varphi_z.$$

Eq. (2) can now be integrated to yield the dynamic boundary condition,

$$\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_z^2) = -gh - \frac{1}{\rho}p.$$

The equations for a fluid are non-dimensionalized by introducing $x = l_x x'$, $z = h_0 z'$, $t = (l_x/c_0)t'$, $\eta = a\eta'$ and $\varphi = (gl_x a/c_0)\varphi'$, where $c_0 = \sqrt{gh_0}$. Being interested in weakly nonlinear, small-amplitude waves in a shallow water environment, we introduce the small parameters $\epsilon = a/h_0$ and $\delta^2 = (h_0/l_x)^2$ where $\epsilon \geq \delta^2 > \epsilon^2 \geq \epsilon\delta^2 \geq \delta^4$. When we talk about the linear approximation we lump together the terms or order ϵ and δ^2, because they are allowed to be of the same order. Similarly, quadratic order means quadratic in ϵ and δ^2, hence includes terms of order ϵ^2, $\epsilon\delta^2$, δ^4. Contrary to this convention the traditional designations third- or fifth-order KdV refer to the power of δ, and hence also the maximal number of derivatives in the linear terms. Upon omitting the primes and expanding the pressure term up to order $\epsilon^2\delta^2$, the Euler equations and the boundary conditions at the free surface and at the bottom are expressed as,

$$\delta^2\varphi_{xx} + \varphi_{zz} = 0 \quad \text{in} \ -1 < z < \epsilon\eta, \tag{4}$$

$$\eta_t + \epsilon\varphi_x\eta_x - \frac{1}{\delta^2}\varphi_z = 0 \quad \text{at} \ z = \epsilon\eta, \tag{5}$$

$$\eta + \varphi_t + \frac{1}{2}\left(\epsilon\varphi_x^2 + \frac{\epsilon}{\delta^2}\varphi_z^2\right) - \sigma\delta^2\eta_{xx} = 0 \quad \text{at} \ z = \epsilon\eta, \tag{6}$$

$$\varphi_z = 0 \quad \text{at } z = -1, \tag{7}$$

where $\sigma = \tilde{\sigma}/(h_0 \rho c_0^2)$ is the dimensionless Bond number.

It is well known that variable transformations of the water wave problem yield the following decoupled equation for the elevation, η,

$$\eta_t + \eta_x + \tfrac{3}{2}\epsilon\eta\eta_x + \tfrac{1}{6}\delta^2(1 - 3\sigma)\eta_{xxx} - \tfrac{3}{8}\epsilon^2\eta^2\eta_x + \epsilon\delta^2(\tfrac{1}{24}(23 + 15\sigma)\eta_x\eta_{xx}$$
$$+ \tfrac{1}{12}(5 - 3\sigma)\eta\eta_{xxx}) + \delta^4\tfrac{1}{360}(19 - 30\sigma - 45\sigma^2)\eta_{xxxxx} = 0. \tag{8}$$

The derivation up to this order appears, for example, in [29], or more recently in [23] without surface tension. Related systematic derivations of higher-order longwave equations were derived in [8,31].

3. Transformation to an integrable equation

Before embarking on its derivation, we shall discuss the transformation properties of Eq. (1). First, it is reversible, i.e., it is invariant under the discrete transformation $u(x, t) \to -u(x, -t)$. Eq. (1) is also Galilean *covariant*. That is, it keeps its form under transformations to an arbitrarily moving reference frame. This includes covariance under transforming to a uniformly moving Galilean frame. However, Eq. (1) is not Galilean *invariant*, even assuming that the momentum, m, Galileo-transforms in the same way as velocity, u. In fact, Eq. (1) transforms under

$$t \to t + t_0, \qquad x \to x + x_0 + ct, \qquad u \to u + c + u_0, \qquad m \to m + c + u_0 \tag{9}$$

to the form

$$m_t + um_x + 2u_x m + (c_0 + u_0)m_x + 2u_x(c + u_0) + \Gamma u_{xxx} = 0. \tag{10}$$

Thus, Eq. (1) is invariant under space and time translations (constants x_0 and t_0), covariant under Galilean transforms (constant c), and acquires linear dispersion terms under velocity shifts (constant u_0). The dispersive term $u_0 m_x$ introduced by the constant velocity shift $u_0 \neq 0$ breaks the reversibility of Eq. (1).

Under scaling transformations of x, t and u, the coefficients of Eq. (1) may be changed. However, such scaling leaves the following coefficient ratios invariant,

$$C(u_x u_{xx}) : C(uu_{xxx}) = 2 : 1, \tag{11}$$

$$C(u_{xxt})C(uu_x) : C(uu_{xxx})C(u_t) = 3 : 1, \tag{12}$$

where $C(f)$ stands for the coefficient of f in the scaled equation. It is pertinent to mention that the above ratios are crucial in the integrability of Eq. (1). See [5,12] for discussions of this point.

Eq. (1) will emerge as being asymptotically equivalent to Eq. (8) after two steps. First, we shall perform a near-identity transformation

$$\eta = \eta(u) = u + \epsilon f[u] + \delta^2 g[u], \tag{13}$$

relating the wave elevation and a "velocity-like" quantity, u. One should not regard u as the original fluid velocity, because we will use a non-local term in f which is difficult to interpret in this context. Instead, we consider u as an auxiliary quantity in which the equation becomes particularly simple. The quantity u agrees at leading order with the fluid velocity and it transforms as a velocity under time reversal, spatial reflection and Galilean boosts. To obtain the physically more meaningful quantity η one has to transform back, see below. The functionals f and g in the transformation (13) are to be chosen so that they generate the terms proportional to uu_x, $u_x u_{xx}$, uu_{xxx} and u_{xxx} in Eq. (1). Afterwards, we shall apply the Helmholtz operator $H = 1 - \nu\delta^2\partial_x^2$, which generates the u_{xxt} term. As in [24] the functional $g[u]$ is proportional to u_{xx} and $f[u]$ is a linear combination of u^2 and a non-local term

$u_x \partial^{-1}$, where ∂^{-1} denotes integration over x. Thus, including the parameter ν, there are four coefficients in this KH transformation. These shall be chosen so that Eq. (1) emerges, after a rescaling of u, x and t.

With these choices, (13) becomes the Kodama transformation, which depends on three parameters α_1, α_2 and β,

$$\eta = \eta(u) = u + \epsilon(\alpha_1 u^2 + \alpha_2 u_x \partial^{-1} u) + \delta^2 \beta u_{xx}. \tag{14}$$

Terms of degree n in the expansion parameters ϵ and δ^2 start contributing at degree $n + 1$ in the transformed equation. Therefore, no terms of quadratic order in ϵ and δ^2 are needed in the transformation. Inserting the Kodama transformation (14) into Eq. (8) for the height field η leads to the following terms in asymptotic order:

$$
\begin{aligned}
\mathcal{O}(1): \quad & u_t + u_x, \\
\mathcal{O}(\epsilon): \quad & 2\alpha_1 u u_t + 2\alpha_1 u u_x + \alpha_2(u_{xt}\partial^{-1}u + u_{xx}\partial^{-1}u + u_x\partial^{-1}u_t + u u_x) + \tfrac{3}{2}u u_x, \\
\mathcal{O}(\delta^2): \quad & \beta u_{xxt} + u_{xxx}(\beta + \tfrac{1}{6} - \tfrac{1}{2}\sigma), \\
\mathcal{O}(\epsilon^2): \quad & \tfrac{9}{2}\alpha_1 u^2 u_x + \tfrac{3}{2}\alpha_2(u^2 u_x + u u_{xx}\partial^{-1}u + u_x^2\partial^{-1}u) - \tfrac{3}{8}u^2 u_x, \\
\mathcal{O}(\epsilon\delta^2): \quad & (\tfrac{23}{24} + \tfrac{5}{8}\sigma + \tfrac{1}{3}(3\alpha_1 + 2\alpha_2)(1 - 3\sigma) + \tfrac{3}{2}\beta)u_x u_{xx} + (\tfrac{5}{12} - \tfrac{1}{4}\sigma \\
& + \tfrac{1}{6}(2\alpha_1 + 3\alpha_2)(1 - 3\sigma) + \tfrac{3}{2}\beta)u u_{xxx} + \tfrac{1}{6}\alpha_2(1 - 3\sigma)u_{xxxx}\partial^{-1}u, \\
\mathcal{O}(\delta^4): \quad & (\tfrac{1}{6}\beta(1 - 3\sigma) + \tfrac{1}{360}(19 - 30\sigma - 45\sigma^2))u_{xxxxx}.
\end{aligned}
\tag{15}
$$

As before, we expand the time derivatives to linear order as

$$u_t = -u_x - \tfrac{3}{2}\epsilon u u_x - \tfrac{1}{6}\delta^2(1 - 3\sigma)u_{xxx}, \tag{16}$$

$$u_{xt} = -u_{xx} - \tfrac{3}{2}\epsilon u_x^2 - \tfrac{3}{2}\epsilon u u_{xx} - \tfrac{1}{6}\delta^2(1 - 3\sigma)u_{xxxx}, \qquad u_{xxt} = -u_{xxx} - \tfrac{9}{2}\epsilon u_x u_{xx} - \tfrac{3}{2}\epsilon u u_{xxx}. \tag{17}$$

This expansion generates higher order terms, leading to

$$
\begin{aligned}
\mathcal{O}(1): \quad & u_t + u_x, \\
\mathcal{O}(\epsilon): \quad & \tfrac{3}{2}u u_x, \\
\mathcal{O}(\delta^2): \quad & \tfrac{1}{6}(1 - 3\sigma)u_{xxx},
\end{aligned}
\tag{18}
$$

$$\mathcal{O}(\epsilon^2): \quad (\tfrac{3}{2}\alpha_1 + \tfrac{3}{4}\alpha_2 - \tfrac{3}{8})u^2 u_x, \tag{19}$$

$$\mathcal{O}(\epsilon\delta^2): \quad \tilde{A}u_x u_{xx} + \tilde{B}u u_{xxx}, \tag{20}$$

$$\mathcal{O}(\delta^4): \quad \tfrac{1}{360}(19 - 30\sigma - 45\sigma^2)u_{xxxxx}, \tag{21}$$

where in (20) we defined

$$\tilde{A} = \tfrac{23}{24} + \tfrac{5}{8}\sigma + \tfrac{1}{2}(2\alpha_1 + \alpha_2)(1 - 3\sigma) - 3\beta \quad \text{and} \quad \tilde{B} = \tfrac{5}{12} - \tfrac{1}{4}\sigma + \tfrac{1}{2}\alpha_2(1 - 3\sigma).$$

The first step of the derivation is now complete. In the second step, applying the Helmholtz operator $H = 1 - \nu\delta^2\partial_x^2$ introduces the coefficient ν and creates terms with two more x derivatives. However, the terms of order $\mathcal{O}(\epsilon^2)$ are left unchanged. These terms are proportional to $u^2 u_x$ and they must vanish for Eq. (1) to emerge. Thus, the application of the Helmholtz operator restores the needed u_{xxt} term that had previously been eliminated. Alternatively, the same Eq. (1) can be obtained by splitting the time derivative, that is, by partially substituting the time derivative u_{xxt} in (15) by its asymptotic approximations (17), along the lines of Benjamin et al. [3] in their study of the BBM equation.

The order $\mathcal{O}(\epsilon^2)$ coefficient in expression (19) will vanish, provided the parameters α_1 and α_2 are chosen to satisfy

$$4\alpha_1 + 2\alpha_2 = 1. \tag{22}$$

The order $\mathcal{O}(\delta^4)$ terms receive an additional contribution that arises from applying the Helmholtz operator $H = 1 - \nu\delta^2\partial_x^2$ to the terms of order $\mathcal{O}(\delta^2)$. The entire combination at order $\mathcal{O}(\delta^4)$ must vanish, for the final equation to possess no fifth-order derivative term, u_{xxxxx}. This requirement determines ν as

$$\nu = \frac{1}{60}\frac{19 - 30\sigma - 45\sigma^2}{1 - 3\sigma}. \tag{23}$$

In what follows, we shall consider the coefficient ν to be given by this function of surface tension, σ. Note: this removal of the highest order term was made possible by introducing the additional parameter ν via the Helmholtz operator. The remaining terms containing free parameters α_2 and β are of order $\epsilon\delta^2$ and they combine additively as

$$(\tilde{A} - \tfrac{9}{2}\nu)u_x u_{xx} + (\tilde{B} - \tfrac{3}{2}\nu)uu_{xxx}.$$

To ensure equivalence to (1) except for scaling we need the relative coefficients to appear in the ratio (11), so that

$$(\tilde{A} - \tfrac{9}{2}\nu) : (\tilde{B} - \tfrac{3}{2}\nu) = 2 : 1. \tag{24}$$

In addition we also need to satisfy (12), so that

$$\tfrac{3}{2}\nu : (\tilde{B} - \tfrac{3}{2}\nu) = 3 : 1.$$

These two conditions imply $\tilde{B} = 2\nu$ and $\tilde{A} = 11\nu/2$. As a result we finally obtain the equation

$$u_t - \nu\delta^2 u_{xxt} + u_x + \tfrac{3}{2}\epsilon uu_x - \tfrac{1}{2}\epsilon\delta^2\nu(uu_{xxx} + 2u_x u_{xx}) + \delta^2(\tfrac{1}{6} - \nu - \tfrac{1}{2}\sigma)u_{xxx} = 0, \tag{25}$$

which can be rewritten in terms of $m = u - \nu\delta^2 u_{xx}$ as

$$m_t + m_x + \tfrac{1}{2}\epsilon(um_x + 2mu_x) + \delta^2\left(\tfrac{1}{6} - \tfrac{1}{2}\sigma\right)u_{xxx} = 0. \tag{26}$$

Thus, the coefficients in the Kodama transformation (14) that yield this equation are

$$\alpha_1 = \frac{7}{20} - \sigma\frac{1}{5}\frac{2 - 3\sigma}{(1 - 3\sigma)^2}, \tag{27}$$

$$\alpha_2 = -\frac{1}{5} + \sigma\frac{2}{5}\frac{2 - 3\sigma}{(1 - 3\sigma)^2}, \tag{28}$$

$$\beta = \frac{1}{30} - \sigma\frac{1}{30}\frac{17 - 30\sigma}{1 - 3\sigma}. \tag{29}$$

Returning to physical variables, where u and φ_x have units of $ga/c_0 = c_0a/h_0$, followed by an additional scaling of $u \to 2u$, gives Eq. (26) the canonical CH form,

$$m_t + c_0 m_x + um_x + 2mu_x + \Gamma u_{xxx} = 0, \tag{30}$$

in which $m = u - \nu h^2 u_{xx}$ and $\Gamma = c_0 h^2(1 - 3\sigma)/6$. The parameters α^2 and Γ in CH (1) are now understood in terms of physical variables as

$$\alpha^2 = \nu h^2 = h^2\frac{1}{60}\frac{19 - 30\sigma - 45\sigma^2}{1 - 3\sigma}, \qquad \Gamma = \frac{c_0 h^2}{6}(1 - 3\sigma). \tag{31}$$

The parameter Γ changes sign when the Bond number σ crosses the critical value, 1/3. For later reference, we record the value of $\sigma > 0$ for which α^2 vanishes, as

$$\sigma_\alpha = -\tfrac{1}{3} + \tfrac{2}{15}\sqrt{30} \simeq 0.39696 > \tfrac{1}{3}. \tag{32}$$

In the special case $c_0 = \Gamma = 0$, Eq. (30) is called "the peakon equation". This equation supports *peakons* as solitary wave solutions whose derivative is discontinuous at the extremum. These solutions were introduced and discussed

by Camassa and Holm [5]. The peakon equation has many exceptional mathematical properties that arise from its interpretation as geodesic motion in the Euler–Poincaré variational framework, as explained in [21]. However, the peakon equation cannot be derived as a water wave equation in a weakly nonlinear shallow approximation from the Euler equation by the present technique. This is because neither a Galilean transformation, nor an appropriate splitting can eliminate both of the linear dispersive terms in Eq. (30) simultaneously. One is always left with a residual linear dispersion.

Johnson [23] has recently derived the dispersive CH equation (30) as a shallow water wave equation by using the same asymptotic expansion. However, two key steps in the asymptotic derivation are addressed in quite different ways in [23] and here. Firstly, the ratios of the coefficients of the equation must be adjusted to ensure integrability. In [23], this is achieved by using the height at which the velocity potential is evaluated as a free parameter. Instead of the height, we use the free parameters in the Kodama transformation (14) to obtain the desired ratios.

Secondly, the fifth-derivative term u_{xxxxx} of order $\mathcal{O}(\delta^4)$ must be removed. The present approach allows the value of the free parameter ν in the Helmholtz operator to be chosen to cancel out this term. In [23], this term is simply omitted. In Section 4 we shall use our approach to show that the CH equation is asymptotically equivalent to the KdV5 equation, which involves this fifth-order derivative.

In order to compare predictions in terms of physically measurable quantities, the solutions for the velocity-like variable u must be transformed back to the elevation field η by using (14). However, the derivation not only used the transformation (14), it also involved applying the Helmholtz operator. Therefore, one should verify that it is sufficient to simply invert (14). Fortunately, when the inverse transformation $u = u(\eta)$ of the same form as (14) with u and η interchanged is substituted into the final Eq. (26), we find that the coefficients just reverse their signs. We conclude that (1) is equivalent to the shallow water wave equation (8) up to and including terms of order $\mathcal{O}(\delta^4)$.

4. The KH transformation

In the previous section, we transformed the shallow water wave equation (8) into the CH equation (1) by means of a Kodama transformation (14) and an application of the Helmholtz-operator. Now we describe the class of equations that can be derived from the shallow water wave equation (8) by any such sequence of transformations, which we will refer to as KH transformation. This class contains (at least) four integrable equations, one of which is the CH equation. Moreover, all equations in this class are not only related to (8); they are also related to each other by such transformations and are therefore asymptotically equivalent.

Similar transformations have been used by Fokas and Liu [15] that included an additional term of the form xu_t, but no Helmholtz-operator. In this case, the class of equations even includes the (third order) KdV equation. Unfortunately, though, this term is not uniformly bounded, so we shall decline to use it. Its unboundedness is a problem when transforming travelling wave solutions which move towards $x = \pm\infty$. Moreover, use of the term xu_t changes the dispersion relation.

4.1. Invariance of the dispersion relation

The Kodama transformation (14) does not change the dispersion relation. To see this, one may observe that only terms linear in η or its derivatives produce linear terms in the transformed equation. Similarly, applying (14) to nonlinear terms in an equation produces only nonlinear terms in the transformed equation. Therefore, in proving invariance of the linear dispersion relation under (14), we may restrict to a transformation $\eta = u + \varepsilon L(u)$ in which L is a linear differential operator with constant coefficients and the linear equation to be transformed is $\eta_t = M(\eta)$. To first order, we then have $u_t = M(u)$ and the full transformation gives

$$u_t + \varepsilon L(u_t) = M(u + \varepsilon L(u)).$$

Now the first-order equation may be used to eliminate the time derivatives which are not of order zero, thereby yielding

$$u_t + \varepsilon L(M(u)) = M(u) + \varepsilon M(L(u)).$$

If M and L commute, as they always do when they have constant coefficients, the final answer is $u_t = M(u)$. Consequently, the Kodama transformation (14) leaves a linear equation unchanged. Note that including the xu_t term in the transformation would in general cause the operators to no longer commute, so that the dispersion relation would be changed, as previously claimed.

The second step of the transformation is the application of the Helmholtz operator $H = 1 - \nu\delta^2\partial_x^2$. As we have just seen, the Kodama transformation leaves the linear part of the equation unchanged. Applying the Helmholtz operator to an equation does change the linear part, but it still leaves the dispersion relation unchanged. To see this, let the linear part of the equation be given by $u_t = M(u)$. The new equation is $H(u_t) = H(M(u))$. If H and M are linear with constant coefficients this gives $H(u)_t = M(H(u))$ so that with the definition $m = H(u)$ we obtain $m_t = M(m)$, which has the same dispersion relation. This does not hold, however, if we truncate higher order terms in $H(M(u))$. If we truncate, then the dispersion relation will agree up to the order of truncation. For example, the dispersion relation for (1) is a rational function, which differs from the polynomial dispersion relation obtained from (8). However, by the above argument the two dispersion relations agree up to the desired order.

4.2. Range of the KH transformation

We now investigate which equations may be transformed into each other by a KH transformation. The class of equations that (under some additional conditions to be derived) may be transformed into each other by KH transformations is given by

$$F(u_t, u_x, u_{xxx}, u_{xxt}, u_{xxxxx}, uu_x) + G(u^2 u_x, uu_{xxx}, u_x u_{xx}) = 0, \tag{33}$$

where F and G are linear and the coefficients of u_t, u_x, uu_x, u_{xxx} are non-zero. This means that each equation has a KdV-kernel. In addition, we assume that the terms are ordered as in the previous section: Every u has weight ϵ and every x (or t) derivative has weight δ. However, for simplicity, we do not explicitly display the weights in the following, even though they are used for truncation at the usual order. The general procedure consists of two parts. In the first part the coefficients in F are normalized in three steps. In the second part the Kodama transformation is used to adjust the terms in G. The details of the calculation are similar to those of the previous section and are therefore not given.

Equations are considered to be equivalent when they differ only by a transformation of the form

$$(t, x, u) \rightarrow (\tau t, \xi x - \kappa t, \nu u). \tag{34}$$

Scaling of u, t, and the equation allow one to set the coefficients of u_x, u_t, and uu_x to arbitrary non-zero values. After such a scaling, the equation takes the form:

$$u_t + u_x + uu_x + f_3 u_{xxx} + f_4 u_{xxt} + f_5 u_{xxxxx} + g_1 u^2 u_x + g_2 uu_{xxx} + g_3 u_x u_{xx} = 0. \tag{35}$$

The next step is to remove the u_{xxt} term by eliminating the t derivatives using the equation itself. This operation produces an equation of the form,

$$u_t + u_x + uu_x + (f_3 - f_4)u_{xxx} + (f_5 - f_4(f_3 - f_4))u_{xxxxx} + g_1 u^2 u_x$$
$$+ (g_2 - f_4)uu_{xxx} + (g_3 - 3f_4)u_x u_{xx} = 0. \tag{36}$$

In the third step, the coefficients of u_{xxx} and u_{xxxxx} are normalized using (34) again, while keeping the coefficients of u_t, u_x, uu_x all fixed at unity. First ξ is used to normalize the ratio between the coefficients of u_{xxx} and u_{xxxxx} to ± 1, assuming that both are non-zero. The relative sign of these coefficients cannot be changed. Dividing the

equation by τ allows both coefficients to be set to unit magnitude. In order to keep the other coefficients in the linear function F equal to unity, the special transformation of the form (34) reads $(t, x, u) \to (\tau t, \xi x + (\tau - \xi)t, \xi u / \tau)$. The above three steps (transformation (34) to find (35), elimination of u_{xxt} to find (36) and again transformation (34) to find (37)) can be performed for any equation in the class (33). Therefore, we shall only be concerned with the equivalence under Kodama transformation of equations of the form

$$u_t + u_x + \tilde{f}_6 u u_x + \tilde{f}_3 u_{xxx} + \tilde{f}_5 u_{xxxxx} + \tilde{g}_1 u^2 u_x + \tilde{g}_2 u u_{xxx} + \tilde{g}_3 u_x u_{xx} = 0, \tag{37}$$

where $\tilde{f}_6 = 1$, $\tilde{f}_3 = 1$, or 0, and $\tilde{f}_5 = \pm 1$, or 0. $\tilde{f}_3 = 0$ appears when $f_3 = f_4$ in (35). The parameter $\tilde{f}_6$ is kept in the notation to trace the influence of the $u u_x$ term. The terms in the linear function F are unchanged by the Kodama transformation because such linear terms are unchanged by a near identity transformation and $u u_x$ is the lowest order nonlinearity. Therefore two equations can only be equivalent if their normalized coefficient $\tilde{f}_5$ agrees. Recall that u_{xxt} has already been eliminated, so that if, e.g., the above steps are applied to the CH equation, the resulting equation does possess a non-zero fifth derivative term.

The terms in the linear function G in (33) can be adjusted by the Kodama transformation. This generates coefficients,

$$\hat{g}_1 = \tfrac{1}{2}(g_1 + \tilde{f}_6(2\alpha_1 + \alpha_2)), \qquad \hat{g}_2 = g_2 + 3\alpha_2 \tilde{f}_3, \qquad \hat{g}_3 = g_3 + 3\tilde{f}_3(2\alpha_1 + \alpha_2) - 2\beta \tilde{f}_6. \tag{38}$$

The $\hat{g}_i$ are linear in the coefficients α_1, α_2, and β of the Kodama transformation. A sufficient condition for the solvability of this linear system is $\tilde{f}_3 \tilde{f}_6^2 \neq 0$. Hence, if the coefficients of u_{xxx} and $u u_x$ in (37) are non-zero, then any value of $\hat{g}_i$ can be achieved by some Kodama transformation. Since the coefficient of $u u_x$ in (33) is assumed to be non-zero the only additional condition is $\tilde{f}_3 = f_3 - f_4 \neq 0$ in (35). Note that this condition was already necessary in order to obtain (37). For example, any equation with positive relative sign of $\tilde{f}_3$ and $\tilde{f}_5$ in (37) is equivalent to the water wave equation (8) with $\sigma = 0$. In particular, this shows that the integrable KdV5 can be obtained, see [27]. Similarly, different higher-order, but non-integrable, extensions of the KdV equation can also be obtained, e.g., those introduced by Champneys and Groves [6]. Notice that the relative sign of $\tilde{f}_3$ and $\tilde{f}_5$ in (37) is negative for $1/3 < \sigma < \sigma_\alpha$, see (32). The condition $\tilde{f}_3 = 0$ appears for $\sigma = 1/3$, while $\tilde{f}_5 = 0$ for $\sigma = \sigma_\alpha \simeq 0.39696$.

To arrive at equations with a u_{xxt} term, the Helmholtz operator is applied. The Helmholtz operator fits nicely into the above procedure, because it is the inverse of the elimination of the u_{xxt} term, at the order considered. More precisely, if the Helmholtz operator $1 - f_4 \partial_x^2$ is applied to (36) then Eq. (35) is recovered. Therefore, each of the three steps leading to the normalized form (37) can also be inverted, and any two equations with equal $\tilde{f}_5$ are asymptotically equivalent. The CH equation has $\tilde{f}_5 = \text{sign}(\alpha^2 \Gamma)$, even though $f_5 = 0$ originally. Moreover, $\tilde{f}_3 = \text{sign}(\Gamma)$, so that the relative sign is that of α^2. From these equations, it follows that the vanishing of $\tilde{f}_3$ implies the vanishing of $\tilde{f}_5$. This is not true in the water wave equation. That is, $\sigma = 1/3$ does not make the coefficient of u_{xxxxx} vanish. This explains why the CH equation is not a good model for $\sigma = 1/3$.

In general, all the terms in the linear function G in Eq. (33) may be removed by a Kodama transformation. However, it is not possible in general at the same time to remove the fifth-order derivative. One may, however, trade the fifth-order derivative for the u_{xxt} term. Therefore, possibly the simplest representative equation in the class (33) has the form:

$$u_t + u_x + u u_x + u_{xxx} + \kappa u_{xxt} = 0,$$

where small positive/negative κ gives the different signs for $\tilde{f}_5$. The case with positive κ results from the BBM equation of Benjamin et al. [3] by a Galilean transformation and velocity shift (which is not included in the transformation group (34)). It is known that the BBM equation is not integrable. Thus, the integrable CH equation arises as a KH near-identity transformation of the non-integrable BBM equation, after a Galilean transformation and velocity shift.

4.3. Examples

4.3.1. Asymptotic equivalence of CH and KdV5 equation

The previous section constructs a transformation from the water wave equation to the CH equation. The general argument of this section shows that it is also possible to transform the CH equation, for example, into the KdV5 equation. To this end, we first expand the time derivative in the u_{xxt}-term using the equation itself and then apply a transformation of the form

$$u = v + \epsilon(\alpha_1 v^2 + \alpha_2 v_x \partial^{-1} v) + \delta^2 \beta v_{xx}. \tag{39}$$

Choosing the values in this Kodama transformation as

$$\alpha_1 = \frac{\alpha^2}{\Gamma}, \qquad \alpha_2 = 2\frac{\alpha^2}{\Gamma}, \qquad \beta = 2\alpha^2, \tag{40}$$

transforms the CH equation (1) into the integrable KdV5 equation

$$v_t + c_0 v_x + 3vv_x + 5(vv_{xxx} + 2v_x v_{xx})\alpha^2 + \frac{15}{2}\frac{\alpha^2 v^2 v_x}{\Gamma} + \Gamma(\alpha^2 v_{xxxxx} + v_{xxx}) = 0. \tag{41}$$

We conclude that (1) is asymptotically equivalent to the integrable KdV5 equation, and both of them are equivalent to (8) at order $\mathcal{O}(\delta^4)$. However, the equivalence of (1) to the KdV5 equation breaks down in the limit $\Gamma \to 0$, because both the transformation and the resulting equation are singular in the limit $\Gamma \to 0$. Therefore, the peakon equation cannot be transformed into KdV5.

Using the additional parameter ν supplied by the Helmholtz operator allows for the removal of the highest order term while preserving the dispersion relation, which is unchanged by applying a linear operator to the equation. One advantage of the CH equation over the asymptotically equivalent KdV5 equation is that it is numerically easier to integrate because it does not contain the fifth derivative. This is in accordance with the general smoothing effect of the Helmholtz operator.

Different variants of higher order KdV equations have been derived in the literature. In this context, we mention [6], who obtained a fifth-order KdV equation by expanding the variational formulation of the water wave problem. Again, their equation is asymptotically equivalent to CH by a certain Kodama transformation. From our point of view, however, the virtue of the power of the Kodama transformation is that it allows one to select integrable equations as the model equation. The equation derived in [6] is not known to be integrable.

4.3.2. Asymptotic equivalence to the b-equation

Recently a new variant of (1) has been introduced by Degasperis and Procesi [9] as

$$m_t + um_x + bu_x m = c_0 u_x - \Gamma u_{xxx}, \tag{42}$$

where b is an arbitrary parameter. The cases $b = 2$ and 3 are special values for this equation. The case $b = 2$ restricts it to the integrable CH equation. The case $b = 3$ is the DP equation of Degasperis and Procesi [9], which was shown to be integrable in [10]. The two cases CH and DP exhaust the integrable candidates for (42), as may be shown using either Painlevé analysis, as in [10], or the asymptotic integrability test, as in [9]. The b-family of equations (42) was also shown in [30] to admit the symmetry conditions necessary for integrability only in the cases $b = 2$ for CH and $b = 3$ for DP.

We shall show here that the new integrable DP equation can also be obtained from the shallow water elevation equation (8) by an appropriate Kodama transformation. The derivation in the previous section is essentially unchanged up to Eq. (24). The two scaling relations (11) and (12) now read

$$(\tilde{A} - \tfrac{9}{2}\nu) : (\tilde{B} - \tfrac{3}{2}\nu) = b : 1, \qquad \tfrac{3}{2}\nu : (\tilde{B} - \tfrac{3}{2}\nu) = b + 1 : 1.$$

These two conditions imply

$$\tilde{B} = \nu \frac{3}{2} \frac{b+2}{b+1} \quad \text{and} \quad \tilde{A} = \nu \frac{3}{2} \frac{4b+3}{b+1}.$$

The resulting Kodama transformation of the form (14) with coefficients α'_1, α'_2, and β' are

$$\alpha'_1 = \alpha_1 + 3\Lambda, \qquad \alpha'_2 = \alpha_2 - 6\Lambda, \qquad \beta' = \beta - (1 - 3\sigma)\Lambda, \quad \text{where} \quad \Lambda = \frac{b-2}{b+1} \frac{45\sigma^2 + 30\sigma - 19}{360}.$$

Therefore, any $b \neq -1$ can be achieved by an appropriate Kodama transformation. Note that when $\sigma = \sigma_\alpha$, see (32), then $\alpha^2 = 0$, hence $\Lambda = 0$ is independent of b. After this transformation (42) is obtained by further scaling the new dependent variable u by the factor $b + 1$. See [22] for discussions of Eq. (42) in which b is treated as a bifurcation parameter when $c_0 = 0$ and $\Gamma = 0$.

The value $b = -1$ is excluded, not from a deficiency of the KH transformation, but because the term uu_x is not removable in the linear function F in Eq. (33). As we have shown above, the KH transformation does not affect this term, and so it cannot be removed from the η equation (8). That is, the case $b = -1$ is not within the range of the KH transformation.

We conclude that the detailed values of the coefficients of the asymptotic analysis at quadratic order hold only modulo the KH transformations, and these transformations may be used to advance the analysis and thereby gain insight. Thus, the KH transformations may provide an answer to the perennial question "Why are integrable equations found so often, when one uses asymptotics in modeling?"

5. Dispersion relation

The interplay between the local and non-local linear dispersion in the CH equation (30) is evident in the relation for its phase velocity,

$$\frac{\omega}{k} = c_0 - \frac{\Gamma k^2}{1 + \alpha^2 k^2} \tag{43}$$

for waves with frequency ω and wave number k linearized around $u = 0$. For $\Gamma < 0$, short waves and long waves both travel in the same direction. Long waves travel faster than short ones (as required in shallow water), provided $\Gamma < 0$. Then the phase velocity lies in $\omega/k \in (c_0 - \Gamma/\alpha^2, c_0]$. At low wave numbers, the constant dispersion parameters α^2 and Γ both perform rather similar functions. At high wave numbers, however, the parameter α^2 properly keeps the phase velocity of the wave from becoming unbounded, and the dispersion relation is similar to the original dispersion relation for water waves, provided that the surface tension vanishes $\sigma = 0$. The remarkably accurate linear dispersion properties close to $k \approx 0$ give the CH equation a clear advantage over the KdV equation (provided σ is not close to 1/3, see Fig. 1).

For the peakon equation, $c_0 = \Gamma = 0$ and linear dispersion is absent. For non-vanishing surface tension, the true dispersion relation for shallow water waves is unbounded for large wave numbers, whereas the dispersion relation of Eq. (1) saturates to the asymptotic value $c_0 - \Gamma/\alpha^2$.

Consequently, (1) is inferior to KdV5 for non-zero surface tension, with respect to travelling wave solutions exhibiting small tail waves. Unboundedness of the linear dispersion relation of KdV5 for high wave numbers allows resonances to occur between a supercritical solitary wave and high-wavenumber linear waves. These resonances give rise to exponentially small ripples at the tails of the solitary wave, in accord with the water wave solution of the full Euler equation. See, for example [4,11,20,28] for more discussion and analysis of these fascinating resonances. Nonetheless, travelling wave solutions of CH and KdV5 are asymptotically equivalent, and they both agree asymptotically with travelling wave solutions of the full water wave problem, see Section 6.

The connections to the physical parameters $\alpha = \alpha(\sigma)$ and $\Gamma = \Gamma(\sigma)$ are defined in equations (31). The asymptotic equation (30) was derived from the water wave equation by means of the KH transformation. As explained in
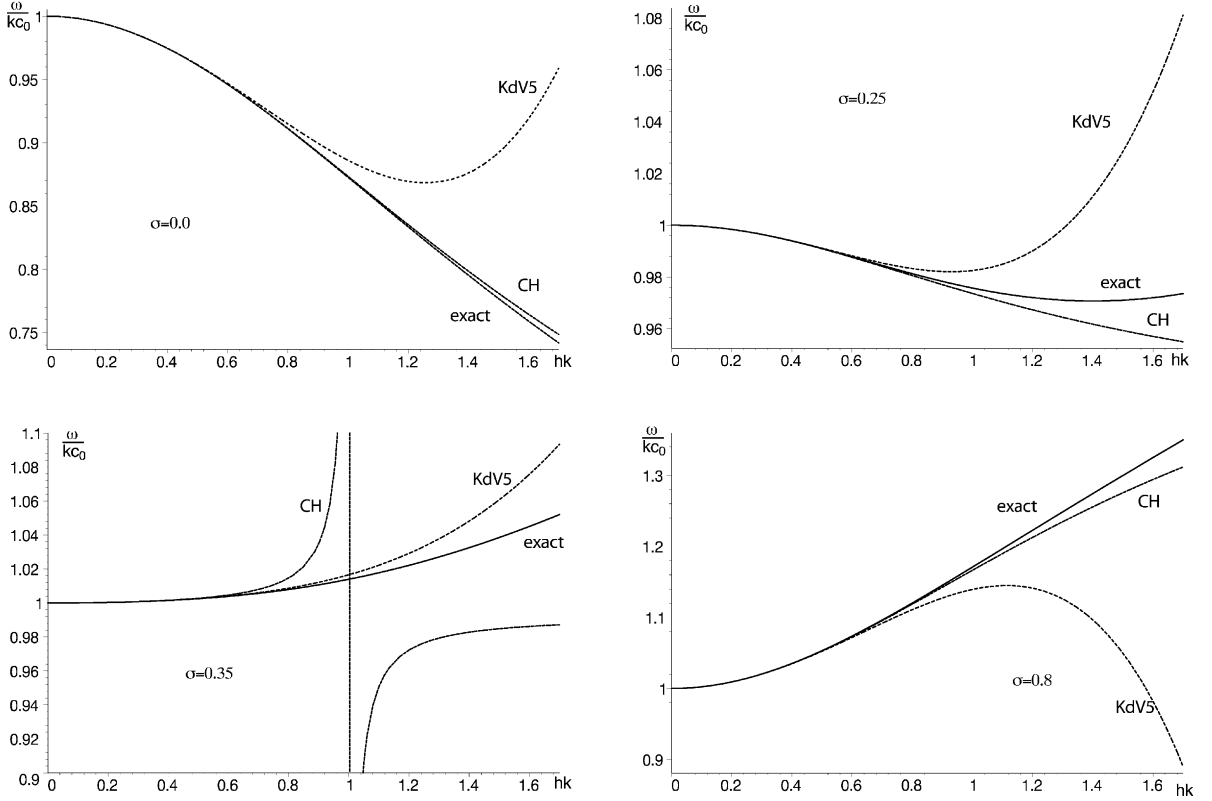
Fig. 1. Normalised dispersion relations for $\sigma = 0.0, 0.25, 0.35, 0.8$ as a function of hk. The full line is the exact phase speed (44); the dashed line is the approximation (43) and the short dashed line is that of KdV5.

Section 4.1, the dispersion relation (43) matches the dispersion relation for water waves up to quintic order. In comparison, the dispersion relation for water waves, when developed for small wave number k yields

$$\frac{\omega}{c_0 k} = \sqrt{\frac{1 + \sigma h^2 k^2}{hk} \tanh hk}, \tag{44}$$

$$\frac{\omega}{c_0 k} \approx 1 - \tfrac{1}{6}(1 - 3\sigma)h^2 k^2 + \tfrac{1}{360}(19 - 30\sigma - 45\sigma^2)h^4 k^4, \tag{45}$$

$$\frac{\omega}{c_0 k} \approx 1 - \tfrac{1}{6}(1 - 3\sigma)h^2 k^2(1 - \nu h^2 k^2). \tag{46}$$

Therefore, the dispersion relations are in agreement up to fifth order in dimensionless wavenumber hk. For the particular value $\sigma \approx 0.069$, the dispersion relations agree up to seventh order.

6. Travelling wave solutions

One may also ask how the solutions of Eq. (1) compare to the usual KdV solitons and to real water solitary waves. An expansion of the form of the travelling wave in the full Euler equations is given by Grimshaw [19] and to higher

order by Fenton [13].[1] Recent results like [28,32] indicate that the problem of existence and uniqueness of solitary waves for the true water wave problem (even with $\sigma = 0$) are very subtle. In the following we merely present a formal argument, and just claim agreement of our soliton solution with the *formal* solution of the Euler equations up to the order ϵa^2. The result given in [19] is normalized so that the highest point of the wave at $s = 0$ is unity.

By direct substitution of a series in sech^2 into (26), one finds

$$u(s) = a \, \text{sech}^2 bs + \tfrac{19}{20} \epsilon a^2 \, \text{sech}^4 bs,$$

where $b^2 = 3a\epsilon/(4\delta^2)$ and $c = 1 + \epsilon a/2 + (19/40)\epsilon^2 a^2$ in $s = x - ct$. Applying the (inverse) Kodama transformation gives

$$\eta(s) = a(1 + \tfrac{1}{2}\epsilon a) \, \text{sech}^2 bs + \tfrac{3}{4}\epsilon a^2 \, \text{sech}^4 bs.$$

For simplicity, we restrict the calculation to the case without surface tension, i.e., $\sigma = 0$. After normalizing the height at the crest to unity, we find perfect agreement up to order ϵa^2 with the exact result by Grimshaw [19]. This shows that the travelling wave solutions of (1) agree with those of the exact Euler equations up to the order we are considering, which is the optimal result. In particular, the travelling waves are narrower and slower (in this normalization) than the KdV soliton, in agreement with experimental findings of Weidman and Maxworthy [34]. The solution also agrees with the one found in [27] by solving the fifth-order KdV equation.

Acknowledgements

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